*Note: accuracy of these solutions cannot be guaranteed – feel free to comment or fix any errors you may see. Also, it is recommended to use the desktop version of Word due to the equations (from the web version, File -> Info).*

## Question 1

### Part 1

Use the second derivative test. Consider the point . The Hessian is then

The eigenvalues (given that the Hessian at this point is a diagonal matrix) are 0 and 2. Given that the Hessian is *not* negative semidefinite at that point, we can be sure that it is *not* concave (since for that to be the case, it will need to be negative semidefinite at all points in ).

So now we look for a counterexample for convexity. Consider the point (1,1). Then, the Hessian is

Solving the problem, we get

Notice that we get a negative eigenvalue, violating the condition which says that the Hessian must be positive semidefinite. Hence, we can confirm that is not convex or concave.

### Part 2

For the FONC part, we want the gradient of *f* to be 0. Then, the goal is to find all points such that and . From the first equation, either or . Then, given , substituting in the second equation, we get . Hence, is the only point satisfying the FONC.

### Part 3

For the SONC part, note that (from part (a))

Let the direction be . Then, computing , which is strictly non-negative for all real *t* and *u*. Hence, it satisfies the SONC.

For the SOSC part, at point (0,0), the Hessian is

This, as shown in part 1, is positive *semidefinite* and not positive *definite* – so it does *not* satisfy the SOSC.

### Part 4

Note that the SOSC cannot be cited to “disprove” the existence of a local minimiser given that it’s only a sufficient condition; that is, there are functions with a local minimiser at a given point that do *not* satisfy the SOSC.

With that out of the way, to show that this one is a saddle point, we need to take two different directions of x and/or y that result in *f* showing different signs at (0,0).

Consider a positive perturbation at *x* (in other words, *x* is increased by a small value *ϵ*). Then, the value of *f* decreases and is negative.

Now consider a positive perturbation at *y* alone (that is, ). Then the value of *f* increases and is positive.

The above two statements show that at (0,0), *f* is neither a local maxima nor a local minima but is a critical point (considering that it satisfies the FONC) -> it’s a saddle point.

### Part 5

**Note**: define the vector .

To start with, find the descent direction . This is , following the gradient result as shown in the lecture.

Then, we want to get . Fit this into *f* to find the minimum:

and find its minimum:

We can show that it’s a minimum by taking the second derivative (which is 8 > 0) or simply looking at the graph.

Finally, finish off the problem by pasting that into the descent equation:

The algorithm terminates here because , as shown in prior parts already.

### Part 6

*Note: the first section of this answer is not necessary – the explanation on why the saddle point won’t be reached is much more important.*

**Note:** as with part 4, define the vector , which is .

Finding the descent direction :

Then, we want to get . Let the vector **P** be defined as – the problem is now to find α which minimises . This happens when . Then,

This is the point of the next iterate – notice that it’s going away from the saddle point.

Now, for an explanation on why we’ll never return to the saddle point. Notice that for that to happen, we *must* get a point such that . But, for that to happen, it must be the case that

Then, this means that *as a necessary condition*

or that *or* . The former is impossible because we aren’t starting at 0 (due to the perturbation). For the latter, this implies that for non-negative , which is also not satisfied.

This confirms that we won’t be returning back to the saddle point at all.

## Question 2

### Part 1

Proof of existence: Since Q is not empty, there exists a point y0 in Q that fits into the argmin and therefore an optimal solution exists.

Proof of uniqueness: Suppose y1 and y2 are different points that both satisfy the projection function such that

Note that this function is strictly convex, there exists a lambda in (0, 1) such that

Here the inequality is strict since the function is strictly convex. Therefore, we have another set of points that is smaller than y1 and y2 (Since the set is a closed set, we can always reach these points) which contradict the assumption, hence prove the uniqueness.

### Part 2

We can rewrite the inner product and expand the second term:

Which gives

Note that the second inequality is derived since x\_0 is not inside Q.

### Part 3

The Lagrangian first order optimality conditions apply when:

First, we compute the Lagrangian and its gradient:

[Vector composed of next four equations]

Then, by the FONC conditions, find points x\* such that .

Notice how there is only a single point, (0,0), that is feasible due to the constraints. However, is the feasible point regular? Let’s find out.

Find the gradient of the constraints:

And at point (0,0), we have

These gradients are *not* linearly independent – an easy way is to get a dependence – one of which is . The lack of regularity of that point is enough to show that the Lagrange first-order optimality conditions do *not* apply for this problem.

### Part 4

The below solution is taken from EdSTEM.

